I. Tabulkové integrály

1. $\int \left(x^4 - 3x^3 + \frac{x^2}{2}\right) \mathrm{d}x = \int x^4 \,\mathrm{d}x - 3\int x^3 \,\mathrm{d}x + \frac{1}{2}\int x^2 \,\mathrm{d}x = \frac{x^5}{5} - \frac{3x^4}{4} + \frac{x^3}{6} + c$

2. $\int \frac{(x-1)^3}{x^2} \mathrm{d}x = \int \frac{x^3 - 3x^2 + 3x - 1}{x^2} \mathrm{d}x = \int \left(x - 3 + \frac{3}{x} - \frac{1}{x^2}\right) \mathrm{d}x = \frac{x^2}{2} - 3x + 3\ln|x| + \frac{1}{x} + c$

3. $\int \frac{(2x-3)^2}{4x^3} \mathrm{d}x = \int \frac{4x^2 - 12x + 9}{4x^3} \mathrm{d}x = \int \left(\frac{1}{x} - \frac{3}{x^2} + \frac{9}{4x^3}\right) \mathrm{d}x = \ln|x| + \frac{3}{x} - \frac{9}{8x^2} + c$

4. $\int (\sqrt{x} - x + 1)(x - \sqrt{x} + 1) \,\mathrm{d}x = \int \left(1 - (\sqrt{x} - x)^2\right) \mathrm{d}x = \int \left(1 - x + 2x^{\frac{3}{2}} - x^2\right) \mathrm{d}x =$

$x - \frac{x^2}{2} + \frac{4x^{\frac{5}{2}}}{5} - \frac{x^3}{3} + c$

5. $\int (\sqrt{x} - 2\sqrt[3]{2x}) \,\mathrm{d}x = \int (x^{\frac{1}{2}} - 2^{\frac{4}{3}} x^{\frac{1}{3}}) \,\mathrm{d}x = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 2^{\frac{4}{3}} \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{2x^{\frac{3}{2}}}{3} - \frac{3x^{\frac{4}{3}}}{2^{\frac{2}{3}}} + c$

6. $\int \left(\frac{2}{\sqrt{x}} - \frac{1}{3\sqrt[3]{x}}\right) \mathrm{d}x = \int \left(2x^{-\frac{1}{2}} - \frac{x^{-\frac{1}{3}}}{3}\right) \mathrm{d}x = 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{1}{3}\frac{x^{\frac{2}{3}}}{\frac{2}{3}} + c = 4x^{\frac{1}{2}} - \frac{x^{\frac{2}{3}}}{2} + c$

7. $\int \frac{(1 + 2\sqrt[3]{x})^2}{2x} \mathrm{d}x = \int \frac{1 + 4x^{\frac{1}{3}} + 4x^{\frac{2}{3}}}{2x} \mathrm{d}x = \int \left(\frac{1}{2x} + 2x^{-\frac{2}{3}} + 2x^{-\frac{1}{3}}\right) \mathrm{d}x = \frac{\ln|x|}{2} + 6x^{\frac{1}{3}} + 3x^{\frac{2}{3}} + c$

8. $\int \frac{x+1}{\sqrt[4]{x^3}} \mathrm{d}x = \int (x^{\frac{1}{4}} + x^{-\frac{3}{4}}) \,\mathrm{d}x = \frac{4x^{\frac{5}{4}}}{5} + 4x^{\frac{1}{4}} + c$

9. $\int \frac{5 - 2x^2}{\sqrt[3]{x^2}} \mathrm{d}x = \int (5x^{-\frac{2}{3}} - 2x^{\frac{4}{3}}) \,\mathrm{d}x = 15x^{\frac{1}{3}} - \frac{12x^{\frac{7}{3}}}{7} + c$

10. $\int \mathrm{tg}^2\, x \,\mathrm{d}x = \int \frac{\sin^2 x}{\cos^2 x} \mathrm{d}x = \int \frac{1 - \cos^2 x}{\cos^2 x} \mathrm{d}x = \int \left(\frac{1}{\cos^2 x} - 1\right) \mathrm{d}x = \mathrm{tg}\, x - x + c$

11. $\int \frac{1}{\sin^2 x \cos^2 x} \mathrm{d}x = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} \mathrm{d}x = \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right) \mathrm{d}x = \mathrm{tg}\, x - \mathrm{cotg}\, x + c$

12. $\int \frac{\cos 2x}{\sin^2 x} \mathrm{d}x = \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x} \mathrm{d}x = \int \frac{1 - 2\sin^2 x}{\sin^2 x} \mathrm{d}x = \int \left(\frac{1}{\sin^2 x} - 2\right) \mathrm{d}x = -\,\mathrm{cotg}\, x - 2x + c$

13. $\int \sin^2 \frac{x}{2} \,\mathrm{d}x = \int \frac{1 - \cos x}{2} \mathrm{d}x = \frac{x - \sin x}{2} + c$

14. $\int \frac{\sin 2x}{2\cos x} \mathrm{d}x = \int \frac{2\sin x \cos x}{2\cos x} \mathrm{d}x = \int \sin x \,\mathrm{d}x = -\cos x + c$

15. $\int \frac{1}{x}(2 - xe^x) \,\mathrm{d}x = \int \left(\frac{2}{x} - e^x\right) \mathrm{d}x = 2\ln|x| - e^x + c$

II. Substituce

1. $\int x(3 - x^2)^2 \,\mathrm{d}x = \frac{1}{2}\int 2x(x^2 - 3)^2 \,\mathrm{d}x = \left| \begin{array}{c} y = x^2 - 3 \\ \mathrm{d}y = 2x \,\mathrm{d}x \end{array} \right| = \frac{1}{2}\int y^2 \,\mathrm{d}y = \frac{y^3}{6} + c = \frac{(x^2-3)^3}{6} + c$

2. $\int x^2\sqrt{1+x^3}\,\mathrm{d}x = \frac{1}{3}\int 3x^2\sqrt{1+x^3}\,\mathrm{d}x = \left|\begin{array}{r} y = 1+x^3 \\ \mathrm{d}y = 3x^2 \end{array}\right| = \frac{1}{3}\int \sqrt{y}\,\mathrm{d}y = \frac{1}{3}\frac{y^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2(1+x^3)^{\frac{3}{2}}}{9} + c$

3. $\int \cos^3 x \sin x\,\mathrm{d}x = \left|\begin{array}{c} y = \cos x \\ \mathrm{d}y = -\sin x\,\mathrm{d}x \end{array}\right| = -\int y^3\,\mathrm{d}y = -\frac{y^4}{4} + c = -\frac{\cos^4 x}{4} + c$

4. $\int \frac{\mathrm{tg}^2 x}{\cos^2 x}\,\mathrm{d}x = \left|\begin{array}{c} y = \mathrm{tg}\, x \\ \mathrm{d}y = \frac{1}{\cos^2 x}\,\mathrm{d}x \end{array}\right| = \int y^2\,\mathrm{d}y = \frac{y^3}{3} + c = \frac{\mathrm{tg}^3 x}{3} + c$

5. $\int e^x \sqrt[3]{e^x - 1}\,\mathrm{d}x = \left|\begin{array}{c} y = e^x - 1 \\ \mathrm{d}y = e^x\,\mathrm{d}x \end{array}\right| = \int \sqrt[3]{y}\,\mathrm{d}y = \frac{y^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{3(e^x-1)^{\frac{4}{3}}}{4} + c$

6. $\int \frac{e^x}{(1+e^x)^2}\,\mathrm{d}x = \left|\begin{array}{c} y = 1+e^x \\ \mathrm{d}y = e^x\,\mathrm{d}x \end{array}\right| = \int \frac{1}{y^2}\,\mathrm{d}y = -\frac{1}{y} + c = -\frac{1}{1+e^x} + c$

7. $\int \frac{\ln^2 x}{x}\,\mathrm{d}x = \left|\begin{array}{c} y = \ln x \\ \mathrm{d}y = \frac{1}{x}\,\mathrm{d}x \end{array}\right| = \int y^2\,\mathrm{d}y = \frac{y^3}{3} + c = \frac{\ln^3 x}{3} + c$

8. $\int \frac{\arcsin x}{\sqrt{1-x^2}}\,\mathrm{d}x = \left|\begin{array}{c} y = \arcsin x \\ \mathrm{d}y = \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \end{array}\right| = \int y\,\mathrm{d}y = \frac{y^2}{2} + c = \frac{\arcsin^2 x}{2} + c$

9. $\int \frac{x}{1+2x^2}\,\mathrm{d}x = \frac{1}{4}\int \frac{4x}{1+2x^2}\,\mathrm{d}x = \left|\begin{array}{c} y = 1+2x^2 \\ \mathrm{d}y = 4x\,\mathrm{d}x \end{array}\right| = \frac{1}{4}\int \frac{1}{y}\,\mathrm{d}y = \frac{\ln|y|}{4} + c = \frac{\ln(1+2x^2)}{4} + c$

10. $\int \frac{1}{x^2}\sin\frac{1}{x}\,\mathrm{d}x = \left|\begin{array}{c} y = \frac{1}{x} \\ \mathrm{d}y = -\frac{1}{x^2}\,\mathrm{d}x \end{array}\right| = -\int \sin y\,\mathrm{d}y = \cos y + c = \cos\frac{1}{x} + c$

11. $\int \frac{\sin x}{\cos^3 x}\,\mathrm{d}x = \left|\begin{array}{c} y = \cos x \\ \mathrm{d}y = -\sin x\,\mathrm{d}x \end{array}\right| = -\int \frac{1}{y^3}\,\mathrm{d}y = -\frac{y^{-2}}{-2} = \frac{1}{2\cos^2 x} + c$

12. $\int \frac{\sin x}{\sqrt[3]{1+2\cos x}}\,\mathrm{d}x = -\frac{1}{2}\int \frac{-2\sin x}{\sqrt[3]{1+2\cos x}}\,\mathrm{d}x = \left|\begin{array}{c} y = 1+2\cos x \\ \mathrm{d}y = -2\sin x\,\mathrm{d}x \end{array}\right| = -\frac{1}{2}\int \frac{1}{\sqrt[3]{y}}\,\mathrm{d}y = -\frac{1}{2}\frac{y^{\frac{2}{3}}}{\frac{2}{3}} + c =$

$-\frac{3(1+2\cos x)^{\frac{2}{3}}}{4} + c$

13. $\int \cos x\sqrt{1-4\sin x}\,\mathrm{d}x = -\frac{1}{4}\int -4\cos x\sqrt{1-4\sin x}\,\mathrm{d}x = \left|\begin{array}{c} y = 1-4\sin x \\ \mathrm{d}y = -4\cos x\,\mathrm{d}x \end{array}\right| = -\frac{1}{4}\int \sqrt{y}\,\mathrm{d}y =$

$-\frac{1}{4}\frac{y^{\frac{3}{2}}}{\frac{3}{2}} + c = -\frac{(1-4\sin x)^{\frac{3}{2}}}{6} + c$

14. $\int \mathrm{tg}\, x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\,\mathrm{d}x = \left|\begin{array}{c} y = \cos x \\ \mathrm{d}y = -\sin x\,\mathrm{d}x \end{array}\right| = -\int \frac{1}{y}\,\mathrm{d}y = -\ln|y| + c = -\ln|\cos x| + c$

15. $\int x e^{-x^2}\,\mathrm{d}x = -\frac{1}{2}\int -2x e^{-x^2}\,\mathrm{d}x = \left|\begin{array}{c} y = -x^2 \\ \mathrm{d}y = -2x\,\mathrm{d}x \end{array}\right| = -\frac{1}{2}\int e^y\,\mathrm{d}y = -\frac{e^y}{2} + c = -\frac{e^{-x^2}}{2} + c$

16. $\int \cos x\, 2^{\sin x}\, \mathrm{d}x = \left| \begin{array}{c} y = \sin x \\ \mathrm{d}y = \cos x\, \mathrm{d}x \end{array} \right| = \int 2^y\, \mathrm{d}y = \frac{2^y}{\ln 2} + c = \frac{2^{\sin x}}{\ln 2} + c$

17. $\int \frac{\sin x}{5 + 3\cos x}\, \mathrm{d}x = -\frac{1}{3} \int \frac{-3\sin x}{5 + 3\cos x}\, \mathrm{d}x = \left| \begin{array}{c} y = 5 + 3\cos x \\ \mathrm{d}y = -3\sin x\, \mathrm{d}x \end{array} \right| = -\frac{1}{3} \int \frac{1}{y}\, \mathrm{d}y = -\frac{\ln|y|}{3} + c =$
$-\frac{\ln|5 + 3\cos x|}{3} + c$

18. $\int \frac{x^3}{2 + x^2}\, \mathrm{d}x = \frac{1}{2} \int \frac{x^2}{2 + x^2}\, 2x\, \mathrm{d}x = \left| \begin{array}{c} y = 2 + x^2 \\ \mathrm{d}y = 2x\, \mathrm{d}x \end{array} \right| = \frac{1}{2} \int \frac{y - 2}{y}\, \mathrm{d}y = \frac{1}{2} \int \left(1 - \frac{2}{y}\right) \mathrm{d}y =$
$\frac{y}{2} - \ln|y| + c = \frac{2 + x^2}{2} - 2\ln(2 + x^2) + c = \frac{x^2}{2} - \ln(2 + x^2) + \tilde{c}$

19. $\int \frac{(1 + \ln x)^2}{x}\, \mathrm{d}x = \left| \begin{array}{c} y = 1 + \ln x \\ \mathrm{d}y = \frac{1}{x}\, \mathrm{d}x \end{array} \right| = \int y^2\, \mathrm{d}y = \frac{y^3}{3} + c = \frac{(1 + \ln x)^3}{3} + c$

20. $\int \frac{1}{x\sqrt[4]{8 - \ln x}}\, \mathrm{d}x = \left| \begin{array}{c} y = 8 - \ln x \\ \mathrm{d}y = -\frac{1}{x}\, \mathrm{d}x \end{array} \right| = -\int \frac{1}{\sqrt[4]{y}}\, \mathrm{d}y = -\frac{y^{\frac{3}{4}}}{\frac{3}{4}} + c = -\frac{4(8 - \ln x)^{\frac{3}{4}}}{3} + c$

21. $\int \frac{3}{(1 - x)^3}\, \mathrm{d}x = \frac{3}{-1} \cdot \frac{(1 - x)^{-2}}{-2} + c = \frac{3}{2(1 - x)^2} + c$

22. $\int \frac{x^2}{x^3 + 1}\, \mathrm{d}x = \frac{1}{3} \int \frac{3x^2}{x^3 + 1}\, \mathrm{d}x = \left| \begin{array}{c} y = x^3 + 1 \\ \mathrm{d}y = 3x^2\, \mathrm{d}x \end{array} \right| = \frac{1}{3} \int \frac{1}{y}\, \mathrm{d}y = \frac{1}{3} \ln|y| + c = \frac{\ln|x^3 + 1|}{3} + c$

23. $\int \frac{e^x}{1 - 2e^x}\, \mathrm{d}x = -\frac{1}{2} \int \frac{-2e^x}{1 - 2e^x}\, \mathrm{d}x = \left| \begin{array}{c} y = 1 - 2e^x \\ \mathrm{d}y = -2e^x\, \mathrm{d}x \end{array} \right| = -\frac{1}{2} \int \frac{1}{y}\, \mathrm{d}y = -\frac{1}{2} \ln|y| + c =$
$-\frac{\ln|1 - 2e^x|}{2} + c$

24. $\int \frac{4}{2 - 3x}\, \mathrm{d}x = 4 \cdot \frac{\ln|2 - 3x|}{-3} + c = -\frac{4\ln|2 - 3x|}{3} + c$

25. $\int \sin(2x + 1)\, \mathrm{d}x = \frac{-\cos(2x + 1)}{2} + c$

26. $\int e^{1-x}\, \mathrm{d}x = \frac{e^{1-x}}{-1} + c = -e^{1-x} + c$

27. $\int \mathrm{tg}\, \frac{x}{2}\, \mathrm{d}x = -2 \int \frac{-\frac{1}{2}\sin\frac{x}{2}}{\cos\frac{x}{2}}\, \mathrm{d}x = \left| \begin{array}{c} y = \cos\frac{x}{2} \\ \mathrm{d}y = -\frac{1}{2}\sin\frac{x}{2}\, \mathrm{d}x \end{array} \right| = -2 \int \frac{1}{y}\, \mathrm{d}y = -2\ln|y| + c = -2\ln\left|\cos\frac{x}{2}\right| + c$

28. $\int \frac{\mathrm{d}x}{9x^2 + 4} = \frac{1}{4} \int \frac{\mathrm{d}x}{\frac{9x^2}{4} + 1} = \frac{1}{4} \int \frac{\mathrm{d}x}{(\frac{3}{2}x)^2 + 1} = \frac{1}{4} \cdot \frac{\mathrm{arctg}\, \frac{3}{2}x}{\frac{3}{2}} + c = \frac{\mathrm{arctg}\, \frac{3}{2}x}{6} + c$

29. $\int 3^{1+2x}\, \mathrm{d}x = \frac{3^{1+2x}}{2\ln 3} + c$

30. $\int \frac{\mathrm{d}x}{\sqrt{1 - 4x^2}} = \int \frac{\mathrm{d}x}{\sqrt{1 - (2x)^2}} = \frac{\arcsin 2x}{2} + c$

III. Per partes

1. $$\int x\ln x\,\mathrm{d}x = \begin{vmatrix} u=\ln x & v'=x \\ u'=\dfrac{1}{x} & v=\dfrac{x^2}{2} \end{vmatrix} = \frac{x^2}{2}\ln x - \int \frac{x}{2}\,\mathrm{d}x = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c = \frac{x^2}{4}(2\ln x - 1) + c$$

2. $$\int \sqrt{x}\ln x\,\mathrm{d}x = \begin{vmatrix} u=\ln x & v'=\sqrt{x} \\ u'=\dfrac{1}{x} & v=\dfrac{2x^{\frac{3}{2}}}{3} \end{vmatrix} = \frac{2x^{\frac{3}{2}}}{3}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,\mathrm{d}x = \frac{2x^{\frac{3}{2}}\ln x}{3} - \frac{4}{9}x^{\frac{3}{2}} + c =$$

$$\frac{2x^{\frac{3}{2}}(3\ln x - 2)}{9} + c$$

3. $$\int \ln x\,\mathrm{d}x = \begin{vmatrix} u=\ln x & v'=1 \\ u'=\dfrac{1}{x} & v=x \end{vmatrix} = x\ln x - \int 1\,\mathrm{d}x = x\ln x - x + c = x(\ln x - 1) + c$$

4. $$\int x\cos x\,\mathrm{d}x = \begin{vmatrix} u=x & v'=\cos x \\ u'=1 & v=\sin x \end{vmatrix} = x\sin x - \int \sin x\,\mathrm{d}x = x\sin x + \cos x + c$$

5. $$\int x^2\cos x\,\mathrm{d}x = \begin{vmatrix} u=x^2 & v'=\cos x \\ u'=2x & v=\sin x \end{vmatrix} = x^2\sin x - 2\int x\sin x\,\mathrm{d}x = \begin{vmatrix} u=x & v'=\sin x \\ u'=1 & v=-\cos x \end{vmatrix} =$$

$$x^2\sin x - 2\left(-x\cos x + \int \cos x\,\mathrm{d}x\right) = x^2\sin x + 2x\cos x - 2\sin x + c$$

6. $$\int x\sin 2x\,\mathrm{d}x = \begin{vmatrix} u=x & v'=\sin 2x \\ u'=1 & v=-\dfrac{\cos 2x}{2} \end{vmatrix} = -\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x\,\mathrm{d}x = -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4} + c$$

7. $$\int x^2e^x\,\mathrm{d}x = \begin{vmatrix} u=x^2 & v'=e^x \\ u'=2x & v=e^x \end{vmatrix} = x^2e^x - 2\int xe^x\,\mathrm{d}x = \begin{vmatrix} u=x & v'=e^x \\ u'=1 & v=e^x \end{vmatrix} =$$

$$x^2e^x - 2\left(xe^x - \int e^x\,\mathrm{d}x\right) = x^2e^x - 2xe^x + 2e^x + c = (x^2 - 2x + 2)e^x + c$$

8. $$\int 2xe^{x+1}\,\mathrm{d}x = \begin{vmatrix} u=x & v'=e^{x+1} \\ u'=1 & v=e^{x+1} \end{vmatrix} = 2\left(xe^{x+1} - \int e^{x+1}\,\mathrm{d}x\right) = 2(xe^{x+1} - e^{x+1}) + c =$$

$$2(x-1)e^{x+1} + c$$

9. $$\int \operatorname{arctg} x\,\mathrm{d}x = \begin{vmatrix} u=\operatorname{arctg} x & v'=1 \\ u'=\dfrac{1}{x^2+1} & v=x \end{vmatrix} = x\operatorname{arctg} x - \int \frac{x}{x^2+1}\,\mathrm{d}x = \begin{vmatrix} y=x^2+1 \\ \mathrm{d}y=2x\,\mathrm{d}x \end{vmatrix} =$$

$$x\operatorname{arctg} x - \frac{1}{2}\int \frac{1}{y}\,\mathrm{d}y = x\operatorname{arctg} x - \frac{\ln|y|}{2} + c = x\operatorname{arctg} x - \frac{\ln(x^2+1)}{2} + c$$

10. $$\int xe^{2x}\,\mathrm{d}x = \begin{vmatrix} u=x & v'=e^{2x} \\ u'=1 & v=\dfrac{e^{2x}}{2} \end{vmatrix} = \frac{xe^{2x}}{2} - \frac{1}{2}\int e^{2x}\,\mathrm{d}x = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c = \frac{e^{2x}}{4}(2x-1) + c$$

11. $$\int \arcsin x\,\mathrm{d}x = \begin{vmatrix} u=\arcsin x & v'=1 \\ u'=\dfrac{1}{\sqrt{1-x^2}} & v=x \end{vmatrix} = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = \begin{vmatrix} y=1-x^2 \\ \mathrm{d}y=-2x\,\mathrm{d}x \end{vmatrix} =$$

$$x \arcsin x + \frac{1}{2}\int \frac{1}{\sqrt{y}}\,\mathrm{d}y = x\arcsin x + \sqrt{y} + c = x \arcsin x + \sqrt{1-x^2} + c$$

12. $$\int \operatorname{arctg}\sqrt{x}\,\mathrm{d}x = \left| \begin{array}{ll} u = \operatorname{arctg}\sqrt{x} & v' = 1 \\ u' = \dfrac{1}{2\sqrt{x}(x+1)} & v = x \end{array} \right| = x \operatorname{arctg}\sqrt{x} - \int \frac{\sqrt{x}}{2(x+1)}\,\mathrm{d}x = \left| \begin{array}{c} y = \sqrt{x} \\ x = y^2 \\ \mathrm{d}x = 2y\,\mathrm{d}y \end{array} \right| =$$

$$x \operatorname{arctg}\sqrt{x} - \int \frac{y}{2(y^2+1)}\,2y\,\mathrm{d}y = x\operatorname{arctg}\sqrt{x} - \int \frac{(y^2+1)-1}{y^2+1}\,\mathrm{d}y =$$

$$x\operatorname{arctg}\sqrt{x} - \int \left(1 - \frac{1}{y^2+1}\right)\mathrm{d}y = x\operatorname{arctg}\sqrt{x} - y + \operatorname{arctg} y + c = (x+1)\operatorname{arctg}\sqrt{x} - \sqrt{x} + c$$

13. $$\int x\ln(x^2+3)\,\mathrm{d}x = \left| \begin{array}{ll} u = \ln(x^2+3) & v' = x \\ u' = \dfrac{2x}{x^2+3} & v = \dfrac{x^2}{2} \end{array} \right| = \frac{x^2\ln(x^2+3)}{2} - \int \frac{x^3}{x^2+3}\,\mathrm{d}x =$$

$$\frac{x^2\ln(x^2+3)}{2} - \frac{1}{2}\int \frac{x^2}{x^2+3}\,2x\,\mathrm{d}x = \left| \begin{array}{c} y = x^2+3 \\ \mathrm{d}y = 2x\,\mathrm{d}x \end{array} \right| = \frac{x^2\ln(x^2+3)}{2} - \frac{1}{2}\int \frac{y-3}{y}\,\mathrm{d}y =$$

$$\frac{x^2\ln(x^2+3)}{2} - \frac{1}{2}\int\left(1 - \frac{3}{y}\right)\mathrm{d}y = \frac{x^2\ln(x^2+3)}{2} - \frac{y}{2} + \frac{3}{2}\ln|y| + c =$$

$$\frac{x^2\ln(x^2+3)}{2} - \frac{x^2+3}{2} + \frac{3}{2}\ln(x^2+3) + c = \frac{(x^2+3)\ln(x^2+3)}{2} - \frac{x^2+3}{2} + c =$$

$$\frac{(x^2+3)(\ln(x^2+3)-1)}{2} + c$$

IV. Racionální funkce

1. $$\int \frac{(x^3-2)^2}{2x^2}\,\mathrm{d}x = \int \frac{x^6 - 4x^3 + 4}{2x^2}\,\mathrm{d}x = \int \frac{x^4}{2} - 2x + \frac{2}{x^2}\,\mathrm{d}x = \frac{x^5}{10} - x^2 - \frac{2}{x} + c$$

2. $$\int \frac{5x+2}{(x+3)^2}\,\mathrm{d}x = \int \frac{5x+15-13}{(x+3)^2}\,\mathrm{d}x = \int \frac{5}{x+3} - \frac{13}{(x+3)^2}\,\mathrm{d}x = 5\ln|x+3| + \frac{13}{x+3} + c$$

3. $$\int \frac{2+x^2}{(2x+1)^3}\,\mathrm{d}x = \int\left(\frac{A}{2x+1} + \frac{B}{(2x+1)^2} + \frac{C}{(2x+1)^3}\right)\mathrm{d}x$$

$$\begin{array}{ll} \multicolumn{2}{l}{A(2x+1)^2 + B(2x+1) + C = 2 + x^2} \\ x = -\frac{1}{2}: & \quad C = \frac{9}{4} \\ x = -1: & A - B + C = 3 \\ x = 0: & \underline{A + B + C = 2} \\ & \underline{A - B \quad = \frac{3}{4}} \\ & \underline{A + B \quad = -\frac{1}{4}} \\ & A = \frac{1}{4},\ B = -\frac{1}{2} \end{array}$$

$$\int\left(\frac{\frac{1}{4}}{2x+1} + \frac{-\frac{1}{2}}{(2x+1)^2} + \frac{\frac{9}{4}}{(2x+1)^3}\right)\mathrm{d}x = \frac{\ln|2x+1|}{8} + \frac{1}{4(2x+1)} - \frac{9}{16(2x+1)^2} + c$$

4. $$\int \frac{\mathrm{d}x}{(3x-4)^4} = \frac{1}{3}\cdot\frac{(3x-4)^{-3}}{-3} + c = -\frac{1}{9(3x-4)^3} + c$$

5. $$\int \frac{x^2-1}{(x+2)^2}\,\mathrm{d}x = \int\left(1 + \frac{-4x-5}{(x+2)^2}\right)\mathrm{d}x = \int\left(1 + \frac{A}{x+2} + \frac{B}{(x+2)^2}\right)\mathrm{d}x$$

$$
\begin{array}{l}
\quad (x^2-1):(x^2+4x+4) = 1 + \dfrac{-4x-5}{x^2+4x+4} \\
\underline{-(x^2+4x+4)} \\
\qquad\quad -4x-5
\end{array}
$$

$$
\begin{array}{ll}
A(x+2)+B=-4x-5 & \\
x=-2: & \quad B = \ \ 3 \\
x=-1: & \underline{A+B=-1} \\
& A=-4
\end{array}
$$

$$\int \left(1 - \frac{4}{x+2} + \frac{3}{(x+2)^2}\right) \mathrm{d}x = x - 4\ln|x+2| - \frac{3}{x+2} + c$$

6. $$\int \frac{x^3}{4-x^2}\,\mathrm{d}x = -\frac{1}{2}\int \frac{x^2}{4-x^2}(-2x)\,\mathrm{d}x = \left| \begin{array}{c} y = 4-x^2 \\ \mathrm{d}y = -2x\,\mathrm{d}x \end{array} \right| = -\frac{1}{2}\int \frac{4-y}{y}\,\mathrm{d}y = -\frac{1}{2}\int \left(\frac{4}{y} - 1\right) \mathrm{d}y =$$

$$-\frac{1}{2}(4\ln|y| - y) + c = -\frac{1}{2}\left(\ln|4-x^2| - (4-x^2)\right) + c = -2\ln|4-x^2| - \frac{x^2}{2} + \tilde{c}$$

7. $$\int \frac{1}{x^2-x-6}\,\mathrm{d}x = \int \frac{1}{(x+2)(x-3)}\,\mathrm{d}x = \int \left(\frac{A}{x+2} + \frac{B}{x-3}\right) \mathrm{d}x$$

$$
\begin{array}{ll}
A(x-3)+B(x+2)=1 & \\
x=-2: & -5A=1 \\
x=3: & \underline{\quad 5B=1 \quad} \\
& A=-\frac{1}{5}, B=\frac{1}{5}
\end{array}
$$

$$\int \left(\frac{-\frac{1}{5}}{x+2} + \frac{\frac{1}{5}}{x-3}\right) \mathrm{d}x = \frac{1}{5}\int \left(\frac{1}{x-3} - \frac{1}{x+2}\right) \mathrm{d}x = \frac{1}{5}(\ln|x-3| - \ln|x+2|) + c =$$

$$\frac{1}{5}\ln\left|\frac{x-3}{x+2}\right| + c$$

8. $$\int \frac{\mathrm{d}x}{x(x+1)} = \int \left(\frac{A}{x} + \frac{B}{x+1}\right) \mathrm{d}x$$

$$
\begin{array}{ll}
A(x+1)+Bx=1 & \\
x=0: & \quad A=1 \\
x=-1: & -B=1
\end{array}
$$

$$\int \left(\frac{1}{x} - \frac{1}{x+1}\right) \mathrm{d}x = \ln|x| - \ln|x+1| + c = \ln\left|\frac{x}{x+1}\right| + c$$

9. $$\int \frac{x+1}{x^2-3x}\,\mathrm{d}x = \int \frac{x+1}{x(x-3)}\,\mathrm{d}x = \int \left(\frac{A}{x} + \frac{B}{x-3}\right) \mathrm{d}x$$

$$
\begin{array}{ll}
A(x-3)+Bx=x+1 & \\
x=0: & -3A=1 \\
x=3: & \underline{\quad 3B=4 \quad} \\
& A=-\frac{1}{3}, B=\frac{4}{3}
\end{array}
$$

$$\int \left(\frac{-\frac{1}{3}}{x} + \frac{\frac{4}{3}}{x-3}\right) \mathrm{d}x = -\frac{1}{3}\ln|x| + \frac{4}{3}\ln|x-3| + c = \frac{1}{3}\ln\frac{(x-3)^4}{|x|} + c$$

10. $$\int \frac{6-3x}{x^2-2x-8}\,\mathrm{d}x = \int \frac{6-3x}{(x-4)(x+2)}\,\mathrm{d}x = \int \left(\frac{A}{x-4} + \frac{B}{x+2}\right) \mathrm{d}x$$

$$
\begin{aligned}
&A(x+2)+B(x-4)=6-3x\\
&x=4: \quad 6A=-6\\
&x=-2: \quad -6B=12\\
&\quad A=-1, B=-2
\end{aligned}
$$

$$\int\left(-\frac{1}{x-4}-\frac{2}{x+2}\right)\mathrm{d}x=-\ln|x-4|-2\ln|x+2|+c$$

11. $$\int\frac{x^2+2}{x^3-2x^2-3x}\,\mathrm{d}x=\int\frac{x^2+2}{x(x+1)(x-3)}\,\mathrm{d}x=\int\left(\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-3}\right)\mathrm{d}x$$

$$
\begin{aligned}
&A(x+1)(x-3)+Bx(x-3)+Cx(x+1)=x^2+2\\
&x=0: \quad -3A=2\\
&x=-1: \quad 4B=3\\
&x=3: \quad 12C=11\\
&\quad A=-\tfrac{2}{3}, B=\tfrac{3}{4}, C=\tfrac{11}{12}
\end{aligned}
$$

$$\int\left(\frac{-\frac{2}{3}}{x}+\frac{\frac{3}{4}}{x+1}+\frac{\frac{11}{12}}{x-3}\right)\mathrm{d}x=-\frac{2}{3}\ln|x|+\frac{3}{4}\ln|x+1|+\frac{11}{12}\ln|x-3|+c$$

12. $$\int\frac{x+1}{x^3-4x}\,\mathrm{d}x=\int\frac{x+1}{x(x+2)(x-2)}\,\mathrm{d}x=\int\left(\frac{A}{x}+\frac{B}{x+2}+\frac{C}{x-2}\right)\mathrm{d}x$$

$$
\begin{aligned}
&A(x+2)(x-2)+Bx(x-2)+Cx(x+2)=x+1\\
&x=0: \quad -4A=1\\
&x=-2: \quad 8B=-1\\
&x=2: \quad 8C=3\\
&\quad A=-\tfrac{1}{4}, B=-\tfrac{1}{8}, C=\tfrac{3}{8}
\end{aligned}
$$

$$\int\left(\frac{-\frac{1}{4}}{x}+\frac{-\frac{1}{8}}{x+2}+\frac{\frac{3}{8}}{x-2}\right)\mathrm{d}x=-\frac{1}{4}\ln|x|-\frac{1}{8}\ln|x+2|+\frac{3}{8}\ln|x-2|+c$$

13. $$\int\frac{\mathrm{d}x}{(x-1)x^2}=\int\left(\frac{A}{x-1}+\frac{B}{x}+\frac{C}{x^2}\right)\mathrm{d}x$$

$$
\begin{aligned}
&Ax^2+B(x-1)x+C(x-1)=1\\
&x=1: \quad A=1\\
&x=0: \quad -C=1\\
&x=2: \quad 4A+2B+C=1\\
&\quad B=-1
\end{aligned}
$$

$$\int\left(\frac{1}{x-1}-\frac{1}{x}-\frac{1}{x^2}\right)\mathrm{d}x=\ln|x-1|-\ln|x|+\frac{1}{x}+c$$

14. $$\int\frac{x-1}{(x+1)(x+2)^2}\,\mathrm{d}x=\int\left(\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}\right)\mathrm{d}x$$

$$
\begin{aligned}
&A(x+2)^2+B(x+1)(x+2)+C(x+1)=x-1\\
&x=-1: \quad A=-2\\
&x=-2: \quad -C=-3\\
&x=0: \quad 4A+2B+C=-1\\
&\quad B=2
\end{aligned}
$$

$$\int\left(\frac{-2}{x+1}+\frac{2}{x+2}+\frac{3}{(x+2)^2}\right)\mathrm{d}x=-2\ln|x+1|+2\ln|x+2|-\frac{3}{x+2}+c$$

15. $\displaystyle\int \frac{3x-4}{(x-2)(x-1)^2}\,\mathrm{d}x = \int \left(\frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\right)\mathrm{d}x$

$$\begin{aligned}
&A(x-1)^2 + B(x-2)(x-1) + C(x-2) = 3x-4\\
&x=2: \quad A = 2\\
&x=1: \quad -C = -1\\
&x=0: \quad A + 2B - 2C = -4\\
&\qquad B = -2
\end{aligned}$$

$$\int \left(\frac{2}{x-2} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right)\mathrm{d}x = 2\ln|x-2| - 2\ln|x-1| - \frac{1}{x-1} + c$$

16. $\displaystyle\int \frac{\mathrm{d}x}{(x+1)(x^2+1)} = \int \left(\frac{A}{x+1} + \frac{Bx+C}{x^2+1}\right)\mathrm{d}x$

$$\begin{aligned}
&A(x^2+1) + (Bx+C)(x+1) = 1\\
&x=-1: \quad 2A = 1\\
&x=0: \quad A + C = 1\\
&x=1: \quad 2A + 2B + 2C = 1\\
&\qquad A = \tfrac{1}{2}, B = -\tfrac{1}{2}, C = \tfrac{1}{2}
\end{aligned}$$

$$\frac{1}{2}\int \left(\frac{1}{x+1} + \frac{1-x}{x^2+1}\right)\mathrm{d}x = \frac{1}{2}\left(\ln|x+1| + \int\left(\frac{1}{x^2+1} - \frac{x}{x^2+1}\right)\mathrm{d}x\right) =$$

$$\frac{1}{2}\left(\ln|x+1| + \operatorname{arctg} x - \frac{1}{2}\int \frac{2x}{x^2+1}\,\mathrm{d}x\right) = \left|\begin{array}{c} y = x^2+1\\ \mathrm{d}y = 2x\,\mathrm{d}x\end{array}\right| = \frac{1}{2}\left(\ln|x+1| + \operatorname{arctg} x - \frac{1}{2}\int\frac{1}{y}\,\mathrm{d}y\right) =$$

$$\frac{1}{2}\left(\ln|x+1| + \operatorname{arctg} x - \frac{\ln|y|}{2}\right) + c = \frac{1}{2}\left(\ln|x+1| + \operatorname{arctg} x - \frac{\ln(x^2+1)}{2}\right) + c$$

17. $\displaystyle\int \frac{x^2-2}{x^2+x-2}\,\mathrm{d}x = \int \left(1 + \frac{-x}{x^2+x-2}\right)\mathrm{d}x = \int\left(1 + \frac{A}{x-1} + \frac{B}{x+2}\right)\mathrm{d}x$

$$\begin{aligned}
&(x^2-2) : (x^2+x-2) = 1 + \frac{-x}{x^2+x-2}\\
&-(x^2+x-2)\\
&\quad -x
\end{aligned}$$

$$\begin{aligned}
&A(x+2) + B(x-1) = -x\\
&x=1: \quad 3A = -1\\
&x=-2: \quad -3B = 2\\
&\qquad A = -\tfrac{1}{3}, B = -\tfrac{2}{3}
\end{aligned}$$

$$\int\left(1 + \frac{-\frac{1}{3}}{x-1} + \frac{-\frac{2}{3}}{x+2}\right)\mathrm{d}x = x - \frac{\ln|x-1|}{3} - \frac{2\ln|x+2|}{3} + c$$

18. $\displaystyle\int \frac{x^3+3}{x^2-3x}\,\mathrm{d}x = \int\left(x + 3 + \frac{9x+3}{x^2-3x}\right)\mathrm{d}x = \int\left(x+3+\frac{A}{x} + \frac{B}{x-3}\right)\mathrm{d}x$

$$\begin{aligned}
&(x^3+3) : (x^2-3x) = x + 3 + \frac{9x+3}{x^2-3x}\\
&-(x^3-3x^2)\\
&\quad 3x^2+3\\
&\quad -(3x^2-9x)\\
&\qquad 9x+3
\end{aligned}$$

$$\begin{aligned}
&A(x-3)+Bx=9x+3\\
&x=0:\ -3A=\ 3\\
&x=3:\ \ 3B=30\\
&\quad A=-1, B=10
\end{aligned}$$

$$\int\left(x+3-\frac{1}{x}+\frac{10}{x-3}\right)\,\mathrm{d}x=\frac{x^2}{2}+3x-\ln|x|+10\ln|x-3|+c$$

19. $$\int\frac{x^3-1}{4x^3-x}\,\mathrm{d}x=\int\left(\frac{1}{4}+\frac{\frac{x}{4}-1}{4x^3-x}\right)\,\mathrm{d}x=\int\left(\frac{1}{4}+\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{2x+1}\right)\,\mathrm{d}x$$

$$\begin{aligned}
&(x^3-1):(4x^3-x)=\tfrac{1}{4}+\frac{\frac{x}{4}-1}{4x^3-x}\\
&-(x^3-\tfrac{x}{4})\\
&\quad \tfrac{x}{4}-1
\end{aligned}$$

$$\begin{aligned}
&A(2x-1)(2x+1)+Bx(2x+1)+Cx(2x-1)=\frac{x}{4}-1\\
&x=0:\quad -A=-1\\
&x=\tfrac{1}{2}:\quad B=-\tfrac{7}{8}\\
&x=-\tfrac{1}{2}:\quad C=-\tfrac{9}{8}
\end{aligned}$$

$$\int\left(\frac{1}{4}+\frac{1}{x}+\frac{-\frac{7}{8}}{2x-1}+\frac{-\frac{9}{8}}{2x+1}\right)\,\mathrm{d}x=\frac{x}{4}+\ln|x|-\frac{7}{16}\ln|2x-1|-\frac{9}{16}\ln|2x+1|+c$$

20. $$\int\frac{x+2}{x^2-2}\,\mathrm{d}x=\int\left(\frac{A}{x-\sqrt{2}}+\frac{B}{x+\sqrt{2}}\right)\,\mathrm{d}x$$

$$\begin{aligned}
&A(x+\sqrt{2})+B(x-\sqrt{2})=x+2\\
&x=\sqrt{2}:\quad 2\sqrt{2}A=\ \sqrt{2}+2\\
&x=-\sqrt{2}:\ -2\sqrt{2}B=-\sqrt{2}+2\\
&\quad A=\tfrac{1+\sqrt{2}}{2}, B=\tfrac{1-\sqrt{2}}{2}
\end{aligned}$$

$$\int\left(\frac{\frac{1+\sqrt{2}}{2}}{x-\sqrt{2}}+\frac{\frac{1-\sqrt{2}}{2}}{x+\sqrt{2}}\right)\,\mathrm{d}x=\frac{1+\sqrt{2}}{2}\ln|x-\sqrt{2}|+\frac{1-\sqrt{2}}{2}\ln|x+\sqrt{2}|+c$$

21. $$\int\frac{3x+1}{x^3-1}\,\mathrm{d}x=\int\left(\frac{A}{x-1}+\frac{B(2x+1)}{x^2+x+1}+\frac{C}{x^2+x+1}\right)\,\mathrm{d}x$$

$$\begin{aligned}
&A(x^2+x+1)+B(2x+1)(x-1)+C(x-1)=3x+1\\
&x=1:\quad 3A=\ 4\\
&x=-\tfrac{1}{2}:\quad \tfrac{3}{4}A-\tfrac{3}{2}C=-\tfrac{1}{2}\\
&x=0:\quad A-\ B-\ C=\ 1\\
&\quad A=\tfrac{4}{3}, B=-\tfrac{2}{3}, C=1
\end{aligned}$$

$$\int\left(\frac{\frac{4}{3}}{x-1}+\frac{-\frac{2}{3}(2x+1)}{x^2+x+1}+\frac{1}{x^2+x+1}\right)\,\mathrm{d}x=\frac{4}{3}\ln|x-1|-\frac{2}{3}\ln(x^2+x+1)+\int\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,\mathrm{d}x=$$

$$\frac{2}{3}\ln\frac{(x-1)^2}{x^2+x+1}+\frac{4}{3}\int\frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\,\mathrm{d}x=\frac{2}{3}\ln\frac{(x-1)^2}{x^2+x+1}+\frac{4}{3}\cdot\frac{\operatorname{arctg}\frac{2x+1}{\sqrt{3}}}{\frac{2}{\sqrt{3}}}+c=$$

$$\frac{2}{3}\ln\frac{(x-1)^2}{x^2+x+1}+\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}+c$$

V. Iracionální funkce

1. $$\int \frac{1}{x\sqrt{x-4}}\,\mathrm{d}x = \left| \begin{array}{c} y=\sqrt{x-4} \\ x=y^2+4 \\ \mathrm{d}x = 2y\,\mathrm{d}y \end{array} \right| = \int \frac{1}{(y^2+4)y}\,2y\,\mathrm{d}y = 2\int \frac{1}{y^2+4}\,\mathrm{d}y = \frac{1}{2}\int \frac{1}{\left(\frac{y}{2}\right)^2+1}\,\mathrm{d}y =$$

$$\frac{1}{2}\frac{\operatorname{arctg}\frac{y}{2}}{\frac{1}{2}} + c = \operatorname{arctg}\frac{\sqrt{x-4}}{2} + c$$

2. $$\int \frac{\mathrm{d}x}{(\sqrt{x}+1)^3} = \left| \begin{array}{c} y=\sqrt{x}+1 \\ x=(y-1)^2 \\ \mathrm{d}x = 2(y-1)\,\mathrm{d}y \end{array} \right| = \int \frac{1}{y^3}\,2(y-1)\,\mathrm{d}y = 2\int \left(y^{-2}-y^{-3}\right)\,\mathrm{d}y = -\frac{2}{y}+\frac{2}{2y^2}+c =$$

$$\frac{1}{(\sqrt{x}+1)^2} - \frac{2}{\sqrt{x}+1} + c$$

3. $$\int \frac{\mathrm{d}x}{x-\sqrt[3]{x}} = \left| \begin{array}{c} y=\sqrt[3]{x} \\ x=y^3 \\ \mathrm{d}x = 3y^2\,\mathrm{d}y \end{array} \right| = \int \frac{1}{y^3-y}\,3y^2\,\mathrm{d}y = \frac{3}{2}\int \frac{2y}{y^2-1}\,\mathrm{d}y = \left| \begin{array}{c} t=y^2-1 \\ \mathrm{d}t = 2y\,\mathrm{d}y \end{array} \right| = \frac{3}{2}\int \frac{1}{t}\,\mathrm{d}t =$$

$$\frac{3}{2}\ln|t| + c = \frac{3}{2}\ln|y^2-1| + c = \frac{3}{2}\ln|x^{\frac{2}{3}}-1| + c$$

4. $$\int \frac{1+\sqrt{x}}{1-\sqrt{x}}\,\mathrm{d}x = \left| \begin{array}{c} y=1-\sqrt{x} \\ x=(1-y)^2 \\ \mathrm{d}x = 2(y-1)\,\mathrm{d}y \end{array} \right| = \int \frac{2-y}{y}\,2(y-1)\,\mathrm{d}y = 2\int \frac{3y-y^2-2}{y}\,\mathrm{d}y =$$

$$2\int \left(3-y-\frac{2}{y}\right)\,\mathrm{d}y = 6y-y^2-4\ln|y|+c = 6(1-\sqrt{x})-(1-\sqrt{x})^2-4\ln|1-\sqrt{x}|+c =$$

$$6-6\sqrt{x}-(1-2\sqrt{x}+x)-4\ln|1-\sqrt{x}|+c = -x-4\sqrt{x}-\ln(1-\sqrt{x})^4+\tilde{c}$$

5. $$\int \frac{x}{(x-1)\sqrt{x-3}}\,\mathrm{d}x = \left| \begin{array}{c} y=\sqrt{x-3} \\ x=y^2+3 \\ \mathrm{d}x = 2y\,\mathrm{d}y \end{array} \right| = \int \frac{y^2+3}{(y^2+2)y}\,2y\,\mathrm{d}y = 2\int \left(1+\frac{1}{y^2+2}\right)\,\mathrm{d}y =$$

$$2y + \int \frac{1}{\left(\frac{y}{\sqrt{2}}\right)^2+1}\,\mathrm{d}y = 2y + \frac{\operatorname{arctg}\frac{y}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} + c = 2\sqrt{x-3}+\sqrt{2}\operatorname{arctg}\sqrt{\frac{x-3}{2}}+c$$

6. $$\int \frac{x^3}{\sqrt{1-x^2}}\,\mathrm{d}x = -\frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,(-2x)\,\mathrm{d}x = \left| \begin{array}{c} y=1-x^2 \\ \mathrm{d}y = -2x\,\mathrm{d}x \end{array} \right| = -\frac{1}{2}\int \frac{1-y}{\sqrt{y}}\,\mathrm{d}y =$$

$$\frac{1}{2}\int \left(\sqrt{y}-\frac{1}{\sqrt{y}}\right)\,\mathrm{d}y = \frac{1}{2}\left(\frac{y^{\frac{3}{2}}}{\frac{3}{2}}-\frac{y^{\frac{1}{2}}}{\frac{1}{2}}\right)+c = \sqrt{y}\left(\frac{y}{3}-1\right)+c = \sqrt{1-x^2}\left(\frac{1-x^2}{3}-1\right)+c =$$

$$-\frac{1}{3}\sqrt{1-x^2}(x^2+2)+c$$

7. $$\int \frac{\sqrt[3]{x}}{1-\sqrt[3]{x}}\,\mathrm{d}x = \left| \begin{array}{c} y=1-\sqrt[3]{x} \\ x=(1-y)^3 \\ \mathrm{d}x = -3(1-y)^2\,\mathrm{d}y \end{array} \right| = \int \frac{1-y}{y}\,\left(-3(1-y)^2\right)\,\mathrm{d}y = 3\int \frac{y^3-3y^2+3y-1}{y}\,\mathrm{d}y =$$

$$3\int \left(y^2-3y+3-\frac{1}{y}\right)\,\mathrm{d}y = y^3-\frac{9}{2}y^2+9y-3\ln|y|+c =$$

$$(1-\sqrt[3]{x})^3-\frac{9}{2}(1-\sqrt[3]{x})^2+9(1-\sqrt[3]{x})-3\ln|1-\sqrt[3]{x}|+c$$

8. $$\int\frac{\sqrt{3+2x}}{x+3}\,\mathrm{d}x=\left|\begin{array}{c} y=\sqrt{3+2x}\\ x=\dfrac{y^2-3}{2}\\ \mathrm{d}x=y\,\mathrm{d}y\end{array}\right|=\int\frac{y}{\frac{y^2-3}{2}+3}y\,\mathrm{d}y=2\int\frac{y^2}{y^2+3}\,\mathrm{d}y=$$

$$2\int\left(\frac{y^2+3}{y^2+3}-\frac{3}{y^2+3}\right)\mathrm{d}y=2\int\left(1-\frac{1}{\left(\frac{y}{\sqrt{3}}\right)^2+1}\right)\mathrm{d}y=2\left(y-\frac{\operatorname{arctg}\frac{y}{\sqrt{3}}}{\frac{1}{\sqrt{3}}}\right)+c=$$

$$2\left(y-\sqrt{3}\operatorname{arctg}\frac{y}{\sqrt{3}}\right)+c=2\left(\sqrt{3+2x}-\sqrt{3}\operatorname{arctg}\sqrt{1+\frac{2}{3}x}\right)+c$$

9. $$\int\frac{\mathrm{d}x}{x(\sqrt[3]{x}-1)}=\left|\begin{array}{c} y=\sqrt[3]{x}-1\\ x=(y+1)^3\\ \mathrm{d}x=3(y+1)^2\,\mathrm{d}y\end{array}\right|=\int\frac{1}{(y+1)^3y}3(y+1)^2\,\mathrm{d}y=3\int\frac{1}{(y+1)y}\,\mathrm{d}y=$$

$$3\int\left(\frac{A}{y}+\frac{B}{y+1}\right)\mathrm{d}y$$

$A(y+1)+By=1$

$y=0:\quad A=1$

$y=-1:\ -B=1$

$$3\int\left(\frac{1}{y}-\frac{1}{y+1}\right)\mathrm{d}y=3(\ln|y|-\ln|y+1|)+c=3\ln\left|1-\frac{1}{\sqrt[3]{x}}\right|+c$$

10. $$\int\frac{\sqrt[6]{x}+1}{\sqrt[6]{x^7}+\sqrt[3]{x^4}}\,\mathrm{d}x=\int\frac{\sqrt[6]{x}+1}{\sqrt[6]{x^7}+\sqrt[6]{x^8}}\,\mathrm{d}x=\int\frac{\sqrt[6]{x}+1}{\sqrt[6]{x^7}(1+\sqrt[6]{x})}\,\mathrm{d}x=\int x^{-\frac{7}{6}}\,\mathrm{d}x=\frac{x^{-\frac{1}{6}}}{-\frac{1}{6}}+c=-\frac{6}{\sqrt[6]{x}}+c$$

11. $$\int\frac{\sqrt[3]{x}}{x+\sqrt[6]{x^5}}\,\mathrm{d}x=\left|\begin{array}{c} y=\sqrt[6]{x}\\ x=y^6\\ \mathrm{d}x=6y^5\,\mathrm{d}y\end{array}\right|=\int\frac{y^2}{y^6+y^5}6y^5\,\mathrm{d}y=6\int\frac{y^2}{y+1}\,\mathrm{d}y=6\int\left(y-1+\frac{1}{y+1}\right)\mathrm{d}y=$$

$$6\left(\frac{y^2}{2}-y+\ln|y+1|\right)+c=3\left(\sqrt[3]{x}-2\sqrt[6]{x}+\ln(\sqrt[6]{x}+1)^2\right)+c$$

12. $$\int x\sqrt[3]{3x+1}\,\mathrm{d}x=\left|\begin{array}{c} y=\sqrt[3]{3x+1}\\ x=\dfrac{y^3-1}{3}\\ \mathrm{d}x=y^2\,\mathrm{d}y\end{array}\right|=\int\frac{y^3-1}{3}y\,y^2\,\mathrm{d}y=\frac{1}{3}\int\left(y^6-y^3\right)\mathrm{d}y=\frac{1}{3}\left(\frac{y^7}{7}-\frac{y^4}{4}\right)+c=$$

$$\frac{1}{84}y^4(4y^3-7)+c=\frac{1}{84}(3x+1)\sqrt[3]{3x+1}\,(4(3x+1)-7)+c=\frac{1}{28}(3x+1)(4x-1)\sqrt[3]{3x+1}+c$$

13. $$\int\sqrt{9-x^2}\,\mathrm{d}x=\left|\begin{array}{c} x=3\sin y\\ \mathrm{d}x=3\cos y\,\mathrm{d}y\\ y=\arcsin\dfrac{x}{3}\end{array}\right|=\int\sqrt{9-9\sin^2 y}\;3\cos y\,\mathrm{d}y=9\int\cos^2 y\,\mathrm{d}y=$$

$$\frac{9}{2}\int(1+\cos 2y)\,\mathrm{d}y=\frac{9}{2}\left(y+\frac{\sin 2y}{2}\right)+c=\frac{9}{2}\left(\arcsin\frac{x}{3}+\sin\left(\arcsin\frac{x}{3}\right)\cos\left(\arcsin\frac{x}{3}\right)\right)+c=$$

$$\frac{9}{2}\left(\arcsin\frac{x}{3}+\frac{x}{3}\sqrt{1-\sin^2\arcsin\frac{x}{3}}\right)+c=\frac{9}{2}\left(\arcsin\frac{x}{3}+\frac{x}{3}\sqrt{1-\left(\frac{x}{3}\right)^2}\right)+c=$$

$$\frac{1}{2}\left(9\arcsin\frac{x}{3}+x\sqrt{9-x^2}\right)+c$$

14. $$\int\frac{\mathrm{d}x}{x^2\sqrt{1-x^2}}=\left|\begin{array}{c}x=\sin y\\ \mathrm{d}x=\cos y\,\mathrm{d}y\\ y=\arcsin x\end{array}\right|=\int\frac{1}{\sin^2 y\sqrt{1-\sin^2 y}}\cos y\,\mathrm{d}y=\int\frac{1}{\sin^2 y}\,\mathrm{d}y=-\operatorname{cotg} y+c=$$

$$-\operatorname{cotg}(\arcsin x)+c=-\frac{\cos(\arcsin x)}{\sin(\arcsin x)}+c=-\frac{\sqrt{1-\sin^2(\arcsin x)}}{x}+c=-\frac{\sqrt{1-x^2}}{x}+c$$

15. $$\int\frac{\sqrt{1-x^2}}{x^2}\,\mathrm{d}x=\left|\begin{array}{c}x=\sin y\\ \mathrm{d}x=\cos y\,\mathrm{d}y\\ y=\arcsin x\end{array}\right|=\int\frac{\sqrt{1-\sin^2 y}}{\sin^2 y}\cos y\,\mathrm{d}y=\int\frac{\cos^2 y}{\sin^2 y}\,\mathrm{d}y=\int\frac{1-\sin^2 y}{\sin^2 y}\,\mathrm{d}y=$$

$$\int\left(\frac{1}{\sin^2 y}-1\right)\mathrm{d}y=-\operatorname{cotg} y-y+c=-\operatorname{cotg}(\arcsin x)-\arcsin x+c=-\frac{\cos(\arcsin x)}{\sin(\arcsin x)}-\arcsin x+c=$$

$$-\frac{\sqrt{1-\sin^2(\arcsin x)}}{x}-\arcsin x+c=-\frac{\sqrt{1-x^2}}{x}-\arcsin x+c$$

VI. Různé integrály

1. $$\int\frac{\cos x}{\sin^2 x+4\sin x}\,\mathrm{d}x=\left|\begin{array}{c}y=\sin x\\ \mathrm{d}y=\cos x\,\mathrm{d}x\end{array}\right|=\int\frac{1}{y^2+4y}\,\mathrm{d}y=\int\left(\frac{A}{y}+\frac{B}{y+4}\right)\mathrm{d}y$$

$$\begin{array}{ll}A(y+4)+By=1 & \\ y=0: & 4A=1\\ y=-4: & -4B=1\\ \hline & A=\frac{1}{4},B=-\frac{1}{4}\end{array}$$

$$\int\left(\frac{\frac{1}{4}}{y}+\frac{-\frac{1}{4}}{y+4}\right)\mathrm{d}y=\frac{1}{4}(\ln|y|-\ln|y+4|)+c=\frac{1}{4}(\ln|\sin x|-\ln|\sin x+4|)+c=\frac{1}{4}\ln\left|\frac{\sin x}{\sin x+4}\right|+c$$

2. $$\int\sin^4 x\,\mathrm{d}x=\int\left(\frac{1-\cos 2x}{2}\right)^2\mathrm{d}x=\frac{1}{4}\int\left(1-2\cos 2x+\cos^2 2x\right)\mathrm{d}x=$$

$$\frac{1}{4}\int\left(1-2\cos 2x+\frac{1-\cos 4x}{2}\right)\mathrm{d}x=\frac{1}{4}\int\left(\frac{3}{2}-2\cos 2x+\frac{\cos 4x}{2}\right)\mathrm{d}x=$$

$$\frac{1}{4}\left(\frac{3}{2}x-\sin 2x+\frac{\sin 4x}{8}\right)+c=\frac{3x}{8}-\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+c$$

3. $$\int\cos^3 x\,\mathrm{d}x=\int(1-\sin^2 x)\cos x\,\mathrm{d}x=\left|\begin{array}{c}y=\sin x\\ \mathrm{d}y=\cos x\,\mathrm{d}x\end{array}\right|=\int(1-y^2)\,\mathrm{d}y=y-\frac{y^3}{3}+c=\sin x-\frac{\sin^3 x}{3}+c$$

4. $$\int\sin^5 x\,\mathrm{d}x=\int(1-\cos^2 x)^2\sin x\,\mathrm{d}x=\left|\begin{array}{c}y=\cos x\\ \mathrm{d}y=-\sin y\,\mathrm{d}x\end{array}\right|=-\int(1-y^2)^2\,\mathrm{d}y=-\int(1-2y^2+y^4)\,\mathrm{d}y=$$

$$-y+\frac{2y^3}{3}+\frac{y^5}{5}+c=-\cos x+\frac{2\cos^3 x}{3}+\frac{\cos^5 x}{5}+c$$

$$5.\ \int \frac{4}{3+5\cos x}\,\mathrm{d}x = \left| \begin{array}{c} y=\operatorname{tg}\dfrac{x}{2} \\ x = 2\operatorname{arctg} y \\ \mathrm{d}x = \dfrac{2}{1+y^2}\,\mathrm{d}y \end{array} \right| = \int \frac{4}{3+5\frac{1-y^2}{1+y^2}}\cdot\frac{2}{1+y^2}\,\mathrm{d}y = \int \frac{8}{8-2y^2}\,\mathrm{d}y = \int\frac{4}{4-y^2}\,\mathrm{d}y =$$

$$\int\left(\frac{A}{2-y}+\frac{B}{2+y}\right)\mathrm{d}y$$

$$\begin{array}{ll} A(2+y)+B(2-y)=4 & \\ y=2: & 4A=4 \\ y=-2: & 4B=4 \\ \hline & A=B=1 \end{array}$$

$$\int\left(\frac{1}{2-y}+\frac{1}{2+y}\right)\mathrm{d}y = -\ln|2-y|+\ln|2+y|+c = \ln\left|\frac{2+y}{2-y}\right|+c = \ln\left|\frac{2+\operatorname{tg}\frac{x}{2}}{2-\operatorname{tg}\frac{x}{2}}\right|+c$$

$$6.\ \int\frac{1}{1-\sin x}\,\mathrm{d}x = \int\frac{1+\sin x}{1-\sin^2 x}\,\mathrm{d}x = \int\left(\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right)\mathrm{d}x = \operatorname{tg} x - \int\frac{-\sin x}{\cos^2 x}\,\mathrm{d}x =$$

$$\left|\begin{array}{c} y=\cos x \\ \mathrm{d}y = -\sin x\,\mathrm{d}x \end{array}\right| = \operatorname{tg} x - \int\frac{1}{y^2}\,\mathrm{d}y = \operatorname{tg} x + \frac{1}{y} + c = \operatorname{tg} x + \frac{1}{\cos x} + c = \frac{\sin x + 1}{\cos x} + c$$

$$7.\ \int\frac{\cos x + 2\sin x}{\cos x - \sin x}\,\mathrm{d}x = \int\frac{\cos x\left(1+2\frac{\sin x}{\cos x}\right)}{\cos x\left(1-\frac{\sin x}{\cos x}\right)}\,\mathrm{d}x = \int\frac{1+2\operatorname{tg} x}{1-\operatorname{tg} x}\,\mathrm{d}x = \left|\begin{array}{c} y = \operatorname{tg} x \\ x = \operatorname{arctg} y \\ \mathrm{d}x = \dfrac{\mathrm{d}y}{1+y^2}\end{array}\right| =$$

$$\int\frac{1+2y}{1-y}\cdot\frac{1}{1+y^2}\,\mathrm{d}y = \int\left(\frac{A}{1-y}+\frac{2By}{1+y^2}+\frac{C}{1+y^2}\right)\mathrm{d}y$$

$$\begin{array}{ll} A(1+y^2)+2By(1-y)+C(1-y)=1+2y & \\ x=1: & 2A = 3 \\ x=0: & A + C = 1 \\ x=-1: & 2A-4B+2C=-1 \\ \hline & A=\frac{3}{2}, B=\frac{3}{4}, C=-\frac{1}{2} \end{array}$$

$$\int\left(\frac{\frac{3}{2}}{1-y}+\frac{\frac{3}{4}\cdot 2y}{1+y^2}+\frac{-\frac{1}{2}}{1+y^2}\right)\mathrm{d}y = -\frac{3}{2}\ln|1-y| + \frac{3}{4}\ln(1+y^2) - \frac{1}{2}\operatorname{arctg} y + c =$$

$$-\frac{3}{2}\ln|1-\operatorname{tg} x| + \frac{3}{4}\ln(1+\operatorname{tg}^2 x) - \frac{1}{2}\operatorname{arctg}\operatorname{tg} x + c = -\frac{3}{2}\ln\left|\frac{\cos x-\sin x}{\cos x}\right| + \frac{3}{4}\ln\frac{\sin^2 x+\cos^2 x}{\cos^2 x} - \frac{x}{2} + c =$$

$$-\frac{3}{2}\ln\left|\frac{\cos x - \sin x}{\cos x}\right| - \frac{3}{2}\ln|\cos x| - \frac{x}{2} + c = -\frac{3}{2}\ln\left(\left|\frac{\cos x-\sin x}{\cos x}\right|\cdot|\cos x|\right) - \frac{x}{2} + c =$$

$$-\frac{3}{2}\ln|\cos x - \sin x| - \frac{x}{2} + c$$

$$8.\ \int\frac{\sin x - 1}{\cos x + 1}\,\mathrm{d}x = \left|\begin{array}{c} y = \operatorname{tg}\dfrac{x}{2} \\ x = 2\operatorname{arctg} y \\ \mathrm{d}x = \dfrac{2}{1+y^2}\,\mathrm{d}y\end{array}\right| = \int\frac{\frac{2y}{1+y^2}-1}{\frac{1-y^2}{1+y^2}+1}\cdot\frac{2}{1+y^2}\,\mathrm{d}y = \int\frac{2y-(1+y^2)}{1+y^2}\,\mathrm{d}y =$$

$$\int\left(\frac{2y}{1+y^2}-1\right)\mathrm{d}y = \left|\begin{array}{c} t = 1+y^2 \\ \mathrm{d}t = 2y\,\mathrm{d}y\end{array}\right| = \int\frac{1}{t}\,\mathrm{d}t - y = \ln|t| - y + c = \ln\left(1+\operatorname{tg}^2\frac{x}{2}\right) - \operatorname{tg}\frac{x}{2} + c =$$

$$\ln \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} - \operatorname{tg} \frac{x}{2} + c = -2 \ln \left|\cos \frac{x}{2}\right| - \operatorname{tg} \frac{x}{2} + c$$

Alternativně:

$$\int \frac{\sin x - 1}{\cos x + 1}\, \mathrm{d}x = \int \left(\frac{\sin x}{\cos x + 1} - \frac{1}{\cos x + 1} \right) \mathrm{d}x = \int \frac{\sin x}{\cos x + 1}\, \mathrm{d}x - \int \frac{1}{2\cos^2 \frac{x}{2}}\, \mathrm{d}x =$$

$$\left| \begin{array}{c} y = \cos x + 1 \\ \mathrm{d}y = -\sin x\, \mathrm{d}x \end{array} \right| = \int \frac{1}{y}\, \mathrm{d}y - \frac{1}{2} \frac{\operatorname{tg} \frac{x}{2}}{\frac{1}{2}} = \ln |y| - \operatorname{tg} \frac{x}{2} = \ln |\cos x + 1| - \operatorname{tg} \frac{x}{2} + c$$

9. $$\int \frac{\cos x}{\cos x - 1}\, \mathrm{d}x = \int \left(\frac{\cos x - 1}{\cos x - 1} + \frac{1}{\cos x - 1} \right) \mathrm{d}x = \int \left(1 - \frac{1}{2\sin^2 \frac{x}{2}} \right) \mathrm{d}x = x + \frac{1}{2} \frac{\operatorname{cotg} \frac{x}{2}}{\frac{1}{2}} + c =$$

$$x + \operatorname{cotg} \frac{x}{2} + c$$

10. $$\int \frac{e^{3x}}{e^x + 1}\, \mathrm{d}x = \left| \begin{array}{c} y = e^x + 1 \\ \mathrm{d}y = e^x\, \mathrm{d}x \end{array} \right| = \int \frac{(y-1)^2}{y}\, \mathrm{d}y = \int \frac{y^2 - 2y + 1}{y}\, \mathrm{d}y = \int \left(y - 2 + \frac{1}{y} \right) \mathrm{d}y =$$

$$\frac{y^2}{2} - 2y + \ln |y| + c = \frac{(e^x + 1)^2}{2} - 2(e^x + 1) + \ln |e^x + 1| + c = \frac{e^{2x}}{2} + e^x + \frac{1}{2} - 2e^x - 2 + \ln |e^x + 1| + c =$$

$$\frac{e^{2x}}{2} - e^x + \ln |e^x + 1| + \tilde{c}$$

11. $$\int \frac{e^x}{e^{3x} + 1}\, \mathrm{d}x = \left| \begin{array}{c} y = e^x \\ \mathrm{d}y = e^x\, \mathrm{d}x \end{array} \right| = \int \frac{1}{y^3 + 1}\, \mathrm{d}y = \int \frac{1}{(y+1)(y^2 - y + 1)}\, \mathrm{d}y =$$

$$\int \left(\frac{A}{y+1} + \frac{B(2y-1)}{y^2 - y + 1} + \frac{C}{y^2 - y + 1} \right) \mathrm{d}y$$

$$\begin{array}{ll} A(y^2 - y + 1) + B(2y-1)(y+1) + C(y+1) = 1 & \\ x = -1: & 3A = 1 \\ x = \frac{1}{2}: & \frac{3}{4}A + \frac{3}{2}C = 1 \\ x = 0: & A - \ B + \ C = 1 \\ \hline & A = \frac{1}{3}, B = -\frac{1}{6}, C = \frac{1}{2} \end{array}$$

$$\int \left(\frac{\frac{1}{3}}{y+1} + \frac{-\frac{1}{6}(2y-1)}{y^2 - y + 1} + \frac{\frac{1}{2}}{y^2 - y + 1} \right) \mathrm{d}y = \frac{1}{3} \ln |y+1| - \frac{1}{6} \ln(y^2 - y + 1) + \frac{1}{2} \int \frac{1}{\left(y - \frac{1}{2}\right)^2 + \frac{3}{4}}\, \mathrm{d}y =$$

$$\frac{1}{6} \ln \frac{(y+1)^2}{y^2 - y + 1} + \frac{2}{3} \int \frac{1}{\left(\frac{2y-1}{\sqrt{3}}\right)^2 + 1}\, \mathrm{d}y = \frac{1}{6} \ln \frac{(y+1)^3}{y^3 + 1} + \frac{2}{3} \frac{\operatorname{arctg} \frac{2y-1}{\sqrt{3}}}{\frac{2}{\sqrt{3}}} + c =$$

$$\frac{1}{6} \ln \frac{(y+1)^3}{y^3 + 1} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2y-1}{\sqrt{3}} + c = \frac{1}{6} \ln \frac{(e^x + 1)^3}{e^{3x} + 1} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2e^x - 1}{\sqrt{3}} + c$$

12. $$\int (3 + \ln^2 x)\, \mathrm{d}x = \left| \begin{array}{ll} u = 3 + \ln^2 x & v' = 1 \\ u' = \dfrac{2\ln x}{x} & v = x \end{array} \right| = x(3 + \ln^2 x) - 2 \int \ln x\, \mathrm{d}x = \left| \begin{array}{ll} u = \ln x & v' = 1 \\ u' = \dfrac{1}{x} & v = x \end{array} \right| =$$

$$x(3 + \ln^2 x) - 2 \left(x \ln x - \int 1\, \mathrm{d}x \right) = x(3 + \ln^2 x) - 2x \ln x + 2x + c = x(\ln^2 x - 2\ln x + 5) + c$$